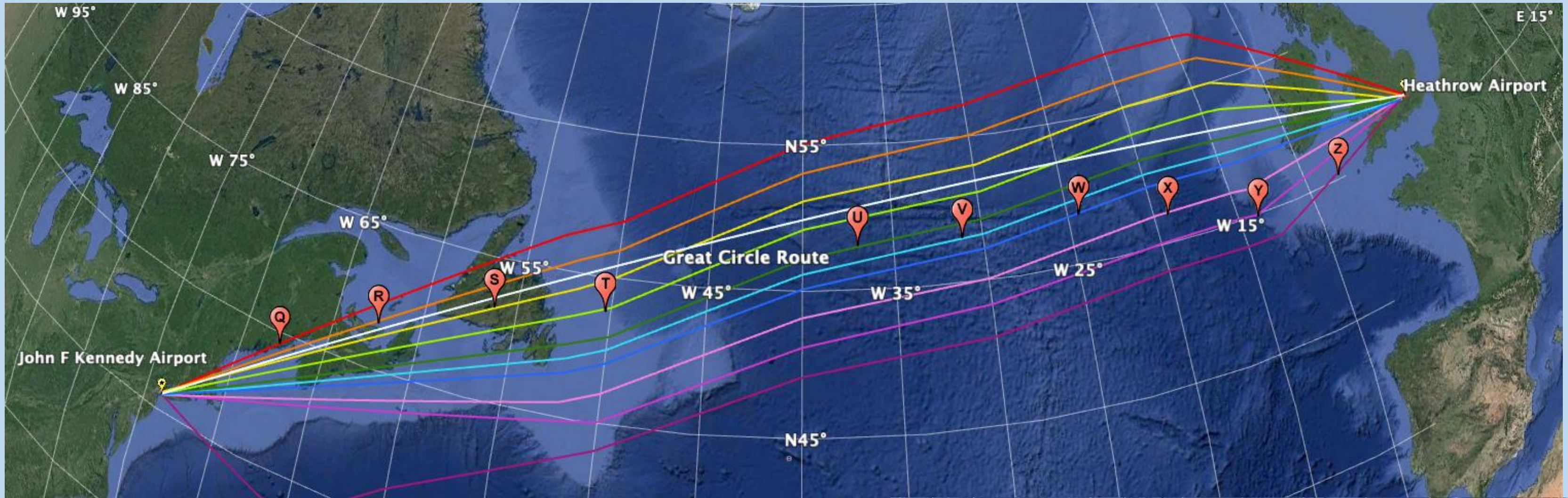
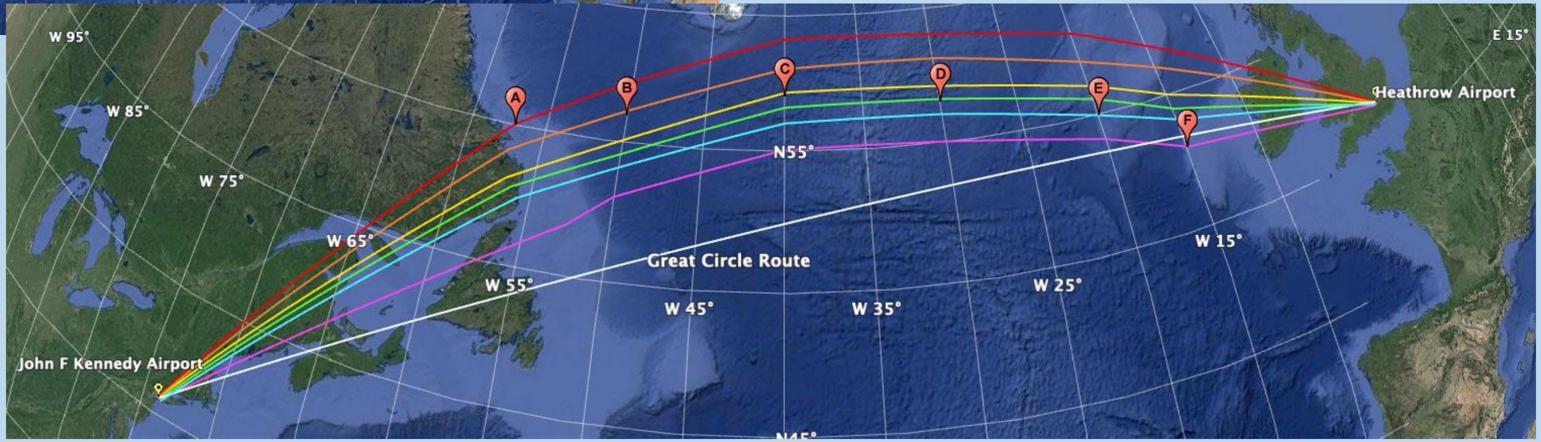
Reducing CO_2 emissions for flights through complex wind fields using three different optimal control approaches.



Cathie Wells

Mathematics of Planet Earth

Map data: Google, Data SIO, NOAA,
U.S. Navy, NGA, GEBCO, Image IBCAO, Image Landsat/Copernicus.



Supervisors:
Paul D. Williams, Nancy K. Nichols,
Dante Kalise, D.I.A. Poll

cathie.wells@reading.ac.uk

Mathematics of Planet Earth

4th April, 2023

Today’s planned route:

- Motivation: if it isn’t broken…
- Time minimal trajectories using an indirect method
- Fixed time trajectories using a direct method
- Free time, fuel minimal trajectories using dynamic programming
- Future horizons

cathie.wells@reading.ac.uk

Motivation: A better future

- $\frac{1}{20}$ of all anthropogenic climate change traceable to the aviation industry. (1)

- 905 million tonnes of CO_2 emitted in 2018. (2)
- ICAO: planes to fly “most fuel efficient route”.(3)
- “If aviation was a country, it would be in the top ten of emitters.”(4)

cathie.wells@reading.ac.uk

Motivation: A better future

- $\frac{1}{20}$ of all ... limate change t ... o the aviation industry. (1)
- 905 ... of CO_2 emitted in 2
- ICAO ... most fuel effici
- “If a ... ountry, it would be in the top ten of emitters.” (4)

61.3 million: metric tonnes of goods carried in 2019. (5)

4.54 billion: passengers carried in 2019. (5)

$691.3 billion: contributed to global GDP in 2019. (6)

>230 000 per day: weather observations recorded. (7)

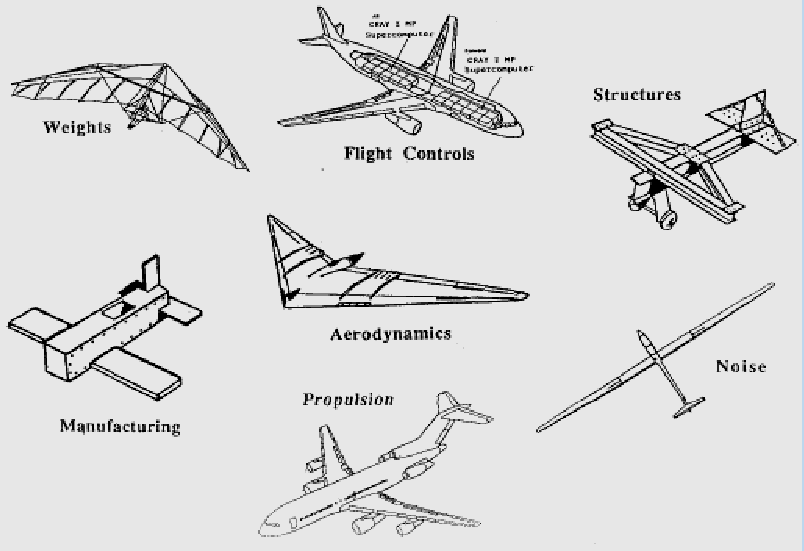
cathie.wells@reading.ac.uk

Alternative ideas

- Change plane design.

  Fleets have become 54% more efficient in the last 30 years.(8)

(a) Turbofan aircraft design (b) Blended-wing-body design



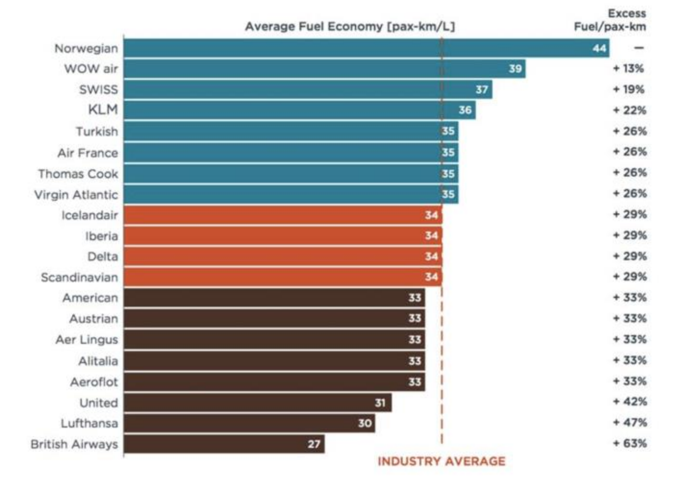
Alternative ideas

- Change plane design.

  Fleets have become 54% more efficient in the last 30 years.(8)

- Put more passengers on each flight.

  Premium passengers: 5.2% of air traffic, but 30.4% of passenger revenue.(8)



Norwegian Named Most Fuel-Efficient Airline on Transatlantic Routes for a Second Time by ICCT

Alternative ideas

- Change plane design.

  Fleets have become 54% more efficient in the last 30 years.(8)

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- Develop flocks of aircraft.

  Fine on paper, but delays would cause fuel usage to be impossible to predict.(9,10)

cathie.wells@reading.ac.uk

Alternative ideas

- Change plane design.

  Fleets have become 54% more efficient in the last 30 years.(8)

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  Premium passengers: 5.2% of air traffic, but 30.4% of passenger revenue.(8)

- Develop flocks of aircraft.

  Fine on paper, but delays would cause fuel usage to be impossible to predict.(9,10)

- Invest in development of biofuels.

  Research is starting now and will still only be able to produce one third of the fuel needed by 2050.(11)

Salicornia bigelovii, aka dwarf saltwort, can grow in saltwater on otherwise unused land, fertilized... [+] GETTY

Motivation: A practical solution

- 100% satellite coverage of North Atlantic.

Photo source: Aireon.

cathie.wells@reading.ac.uk

Motivation: A practical solution

- 100% satellite coverage of North Atlantic.
- Trajectory Based Operations: better efficiency.

Photo source: Airbus.com

cathie.wells@reading.ac.uk

4th April, 2023

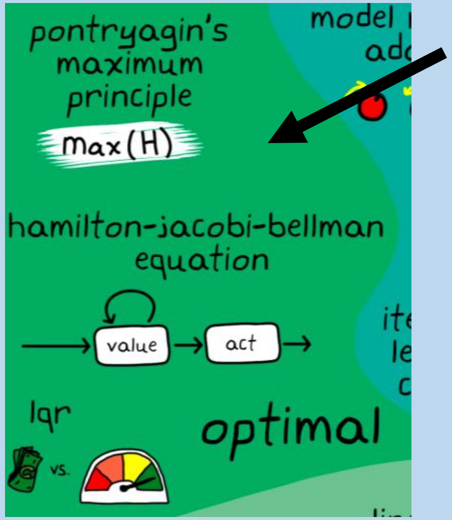
Motivation: A practical solution

- ➢100% satellite coverage of North Atlantic.
- ➢Trajectory Based Operations: better efficiency.
- ➢Airlines need to reduce emissions, whilst adhering to a schedule.

11:40		Paris	Delayed
11:45	RT 4545	Montreal	Estimated 11:45
11:50	QE 0900	New York	Delayed
11:50	QE 0880	Montevideo	On Time
11:55	QE 2105	Barcelona	Delayed
12:00	SQ 0092	Toronto City	On Time
12:00	XZ 0766	Rome	Estimated 12:45
12:00	AK 7304	Sydney	Delayed
12:10	AK 8900	Hong Kong	Delayed
12:15	SQ 0232	Washington D.C.	
12:20	XZ 3170	Tokyo	
12:20	SQ 0080		
12:30			

Photo source: travel.stackexchange.com

Optimal control theory:



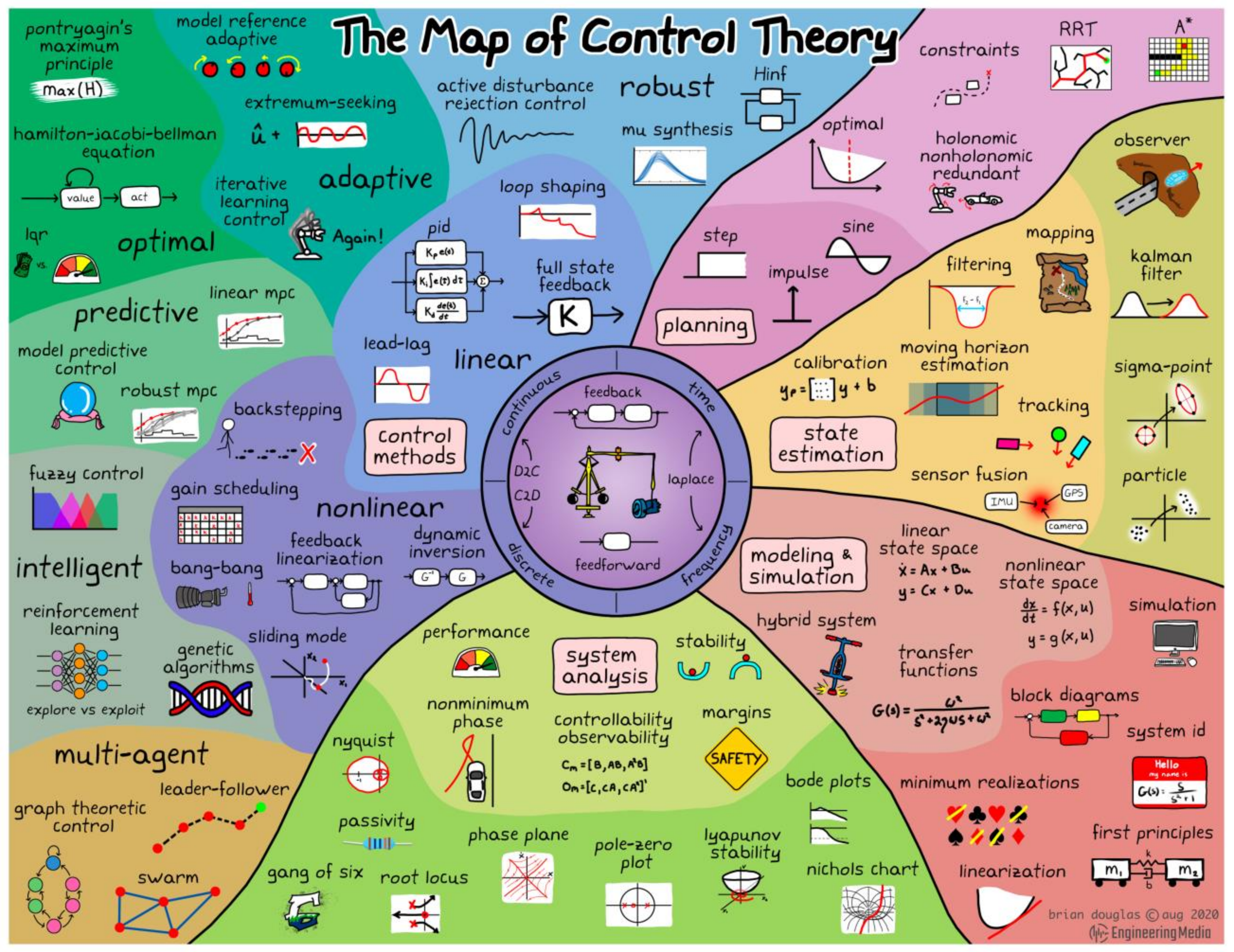


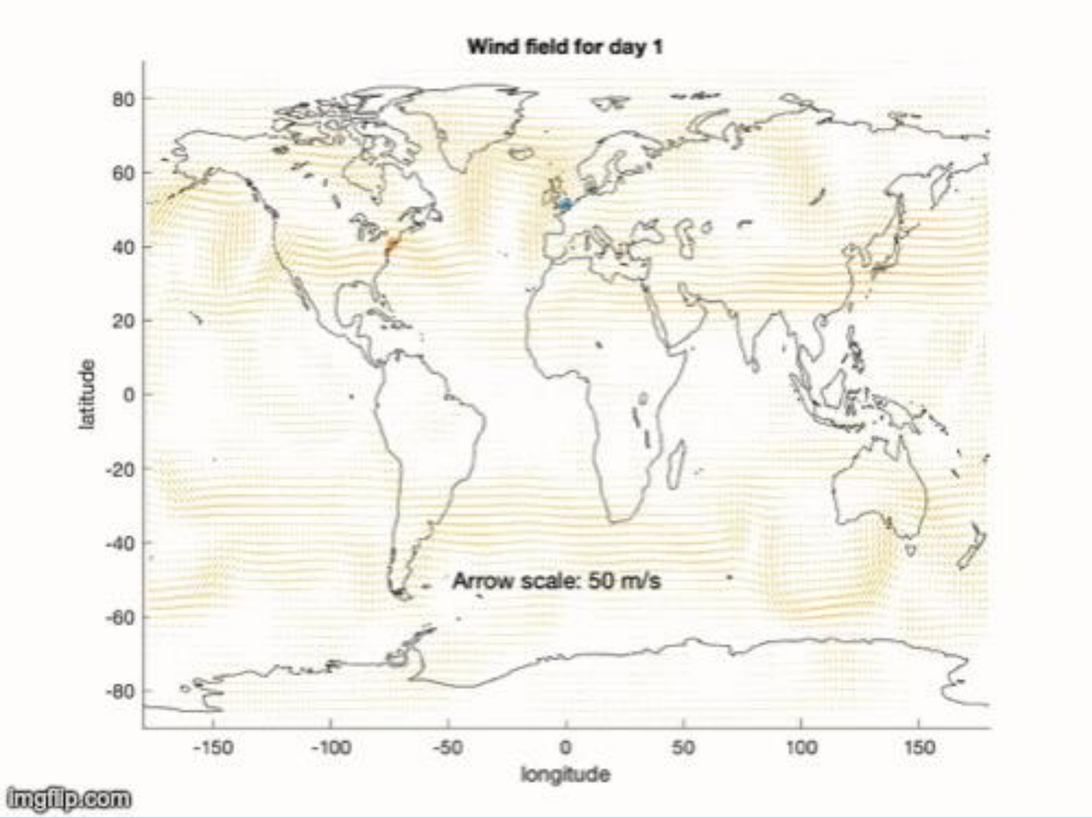
Figure 1: Control Theory Map by Brian Douglas, https://engineeringmedia.com/.

Three main methods:

- Indirect methods
- Direct methods
- Dynamic programming

Time minimal trajectories using indirect methods

How much difference would horizontal, time optimal trajectory planning make to fuel use and thus carbon dioxide emissions, in fixed airspeed transatlantic flights compared with the Organised Track Structure?



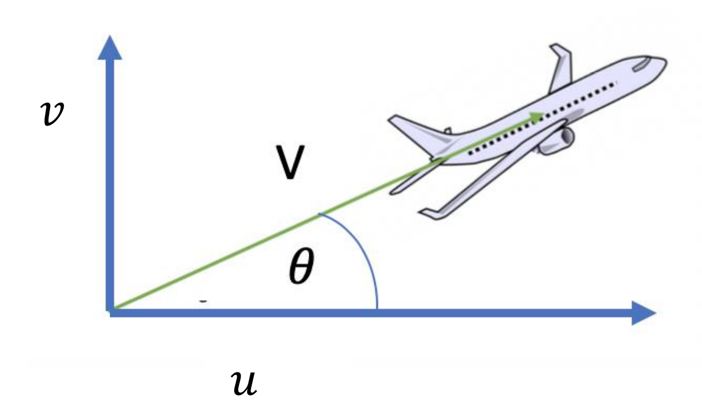
Method:

- ➢Routes: JFK (40.6^{o}N,73.8^{o}W) and LHR (51.5^{o}N,0.5^{o}W).
- ➢Winter period: 1st December, 2019 to 29th February, 2020.
- ➢Winds from: National Center for Atmospheric Research (2.5^{o} resolution).(14)
- ➢Constant altitude FL340 equating approximately to a pressure of 250 hPa.
- ➢Constant airspeed across each trajectory, from 200 m/s to 270 m/s.
- ➢Air distance: airspeed x flight time.
- ➢Fuel usage and emissions proportional to air distance.

cathie.wells@reading.ac.uk

Defining the spherical system

Defining the symbols:



λ	longitude in radians
ϕ	latitude in radians
θ	heading angle in radians
u	zonal wind in m s^{-1}
v	meridional wind in m s^{-1}
V	air speed of aircraft in m s^{-1}
R	radius of Earth in m (here approximated to 6 371 000 m)
t	time in seconds

Defining the problem

Parameters:

$$t_0 = 0$$
$$t_{final} \geq 0$$

Boundary conditions:

$$\lambda(0) = \lambda_{\text{dept}}$$
$$\phi(0) = \phi_{\text{dept}}$$

Dynamical system:

$$\dot{\lambda} = \frac{V\cos\theta + \mathrm{u}(\lambda,\phi)}{R\cos\phi}$$
$$\dot{\phi} = \frac{V\sin\theta + \mathrm{v}(\lambda,\phi)}{R}$$

State variables:

$$x_1 = \lambda(t)$$
$$x_2 = \phi(t)$$

Control variable:

$$\alpha(t) = \theta$$

Cost functional:

$$J(\boldsymbol{x},\alpha) = \int_{t_0}^{t_{final}} 1\,dt = t_{final}$$

Method: Euler Forward Step

$$\frac{d\lambda}{dt} = \frac{V\cos\theta + u(\lambda,\phi)}{R\cos\phi} \quad\Rightarrow\quad lo_{t+1} = lo_t + \frac{u_t + V\cos he_t}{R\cos la_t} \times dt$$

$$\frac{d\phi}{dt} = \frac{V\sin\theta + v(\lambda,\phi)}{R} \quad\Rightarrow\quad la_{t+1} = la_t + \frac{v_t + V\sin he_t}{R} \times\ dt$$

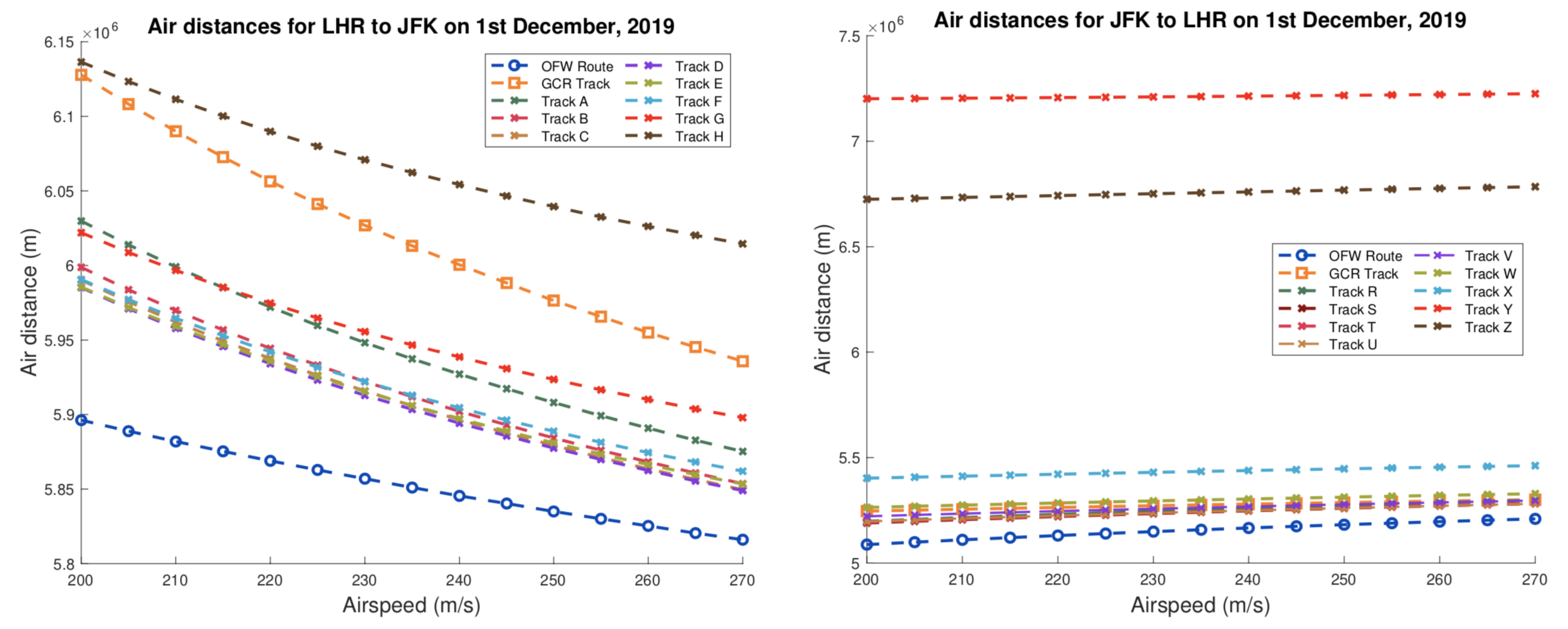
$$\frac{d\theta}{dt} = -\frac{1}{R\cos\phi}\Big[-\sin\theta\cos\theta\frac{\partial u}{\partial\lambda} + u\cos^2\theta\sin\phi + \cos^2\theta\cos\phi\frac{\partial u}{\partial\phi} - \frac{\partial v}{\partial\lambda} + \cos^2\lambda\frac{\partial v}{\partial\lambda}\ldots$$
$$+ v\sin\theta\cos\theta\sin\phi + \sin\theta\cos\theta\cos\phi\frac{\partial v}{\partial\lambda} + V\cos\theta\sin\phi\Big]$$

$$\Rightarrow\quad he_{t+1} = he_t - \frac{Wind\ \times dt}{R\cos la_t}$$

$$Wind = -\sin he_t\cos he_t\left(\frac{du}{dlo}\right)_t + \ u\cos^2 he_t\sin la_t + \cos^2 he_t\cos la_t\left(\frac{du}{dla}\right)_t\ \ldots$$
$$-\left(\frac{dv}{dlo}\right)_t + \cos^2 lo_t\left(\frac{dv}{dlo}\right)_t + v\sin he_t\cos he_t\sin la_t\ldots$$

$$+\sin he_t\cos he_t\cos la_t\left(\frac{dv}{dla}\right)_t + \mathrm{V}\cos he_t\sin la_t$$

Results: Multiple tracks



(a) Air distances 1st December, 2019 westbound

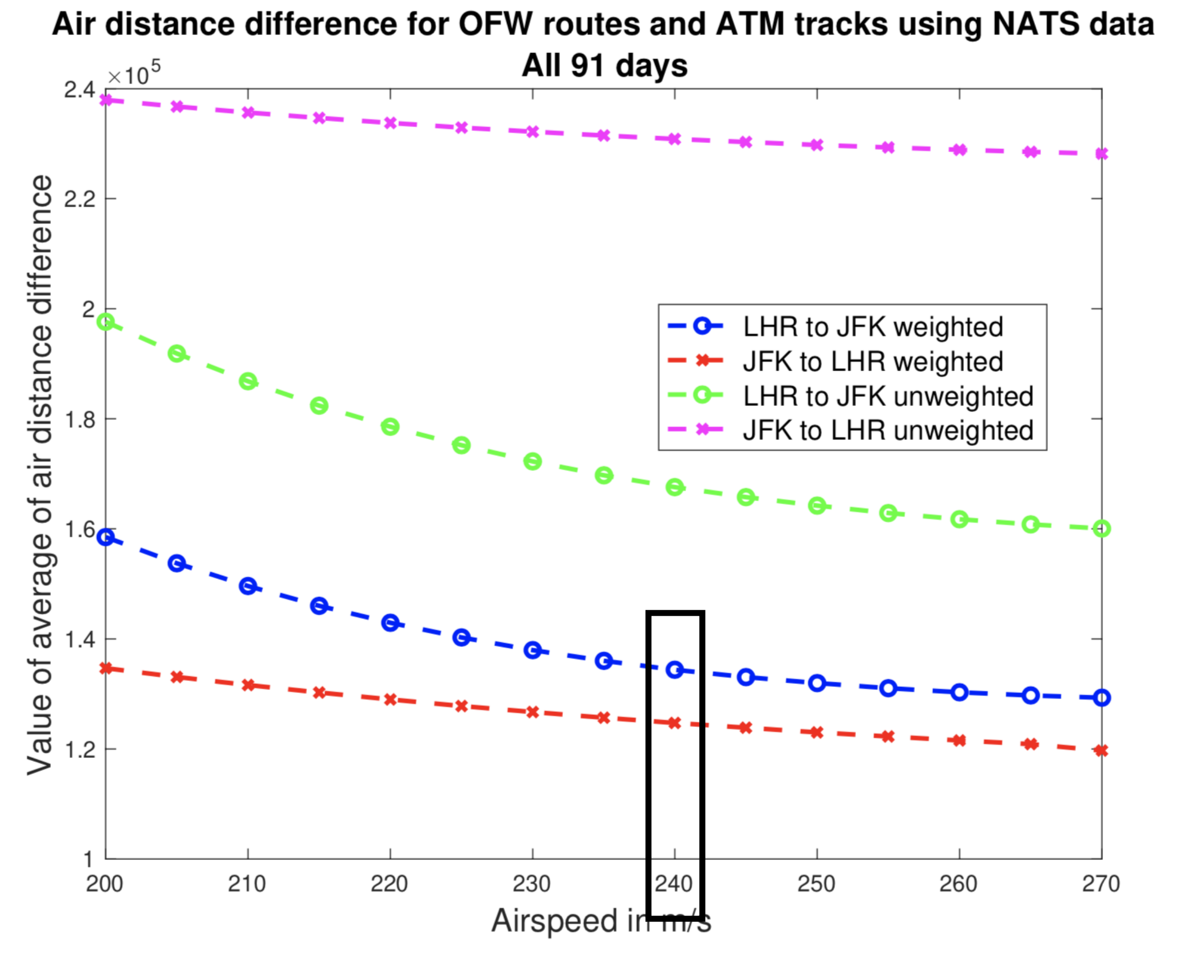
(b) Air distances 1st December, 2019 eastbound

cathie.wells@reading.ac.uk

Mathematics of Planet Earth

4th April, 2023

Results: Savings



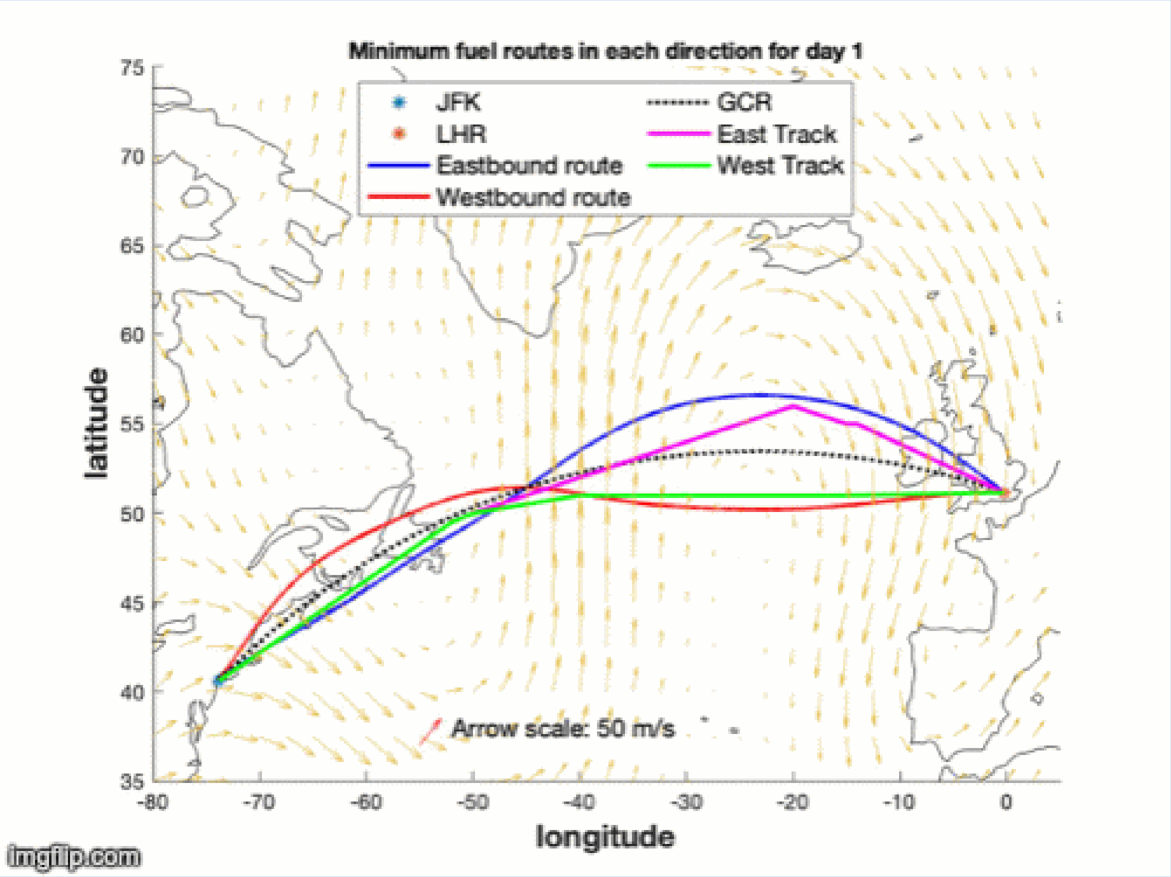
Unweighted data:
Assumes equal numbers of flights along each track supplied by NATS each day. Savings found by taking average air distance change over each track, each day.

Weighted data:
Uses information supplied by NATS to show how many aircraft used each track each day. Savings are found by a weighted average of air distance savings across all tracks each day.

Overview of Findings:

Route	% improvement	
	Best	Worst
JFK to LHR	0.9	16.4
LHR to JFK	1.1	7.8

Route	% improvement at 240 m/s
	Weighted Average
JFK to LHR	2.5
LHR to JFK	1.7



cathie.wells@reading.ac.uk

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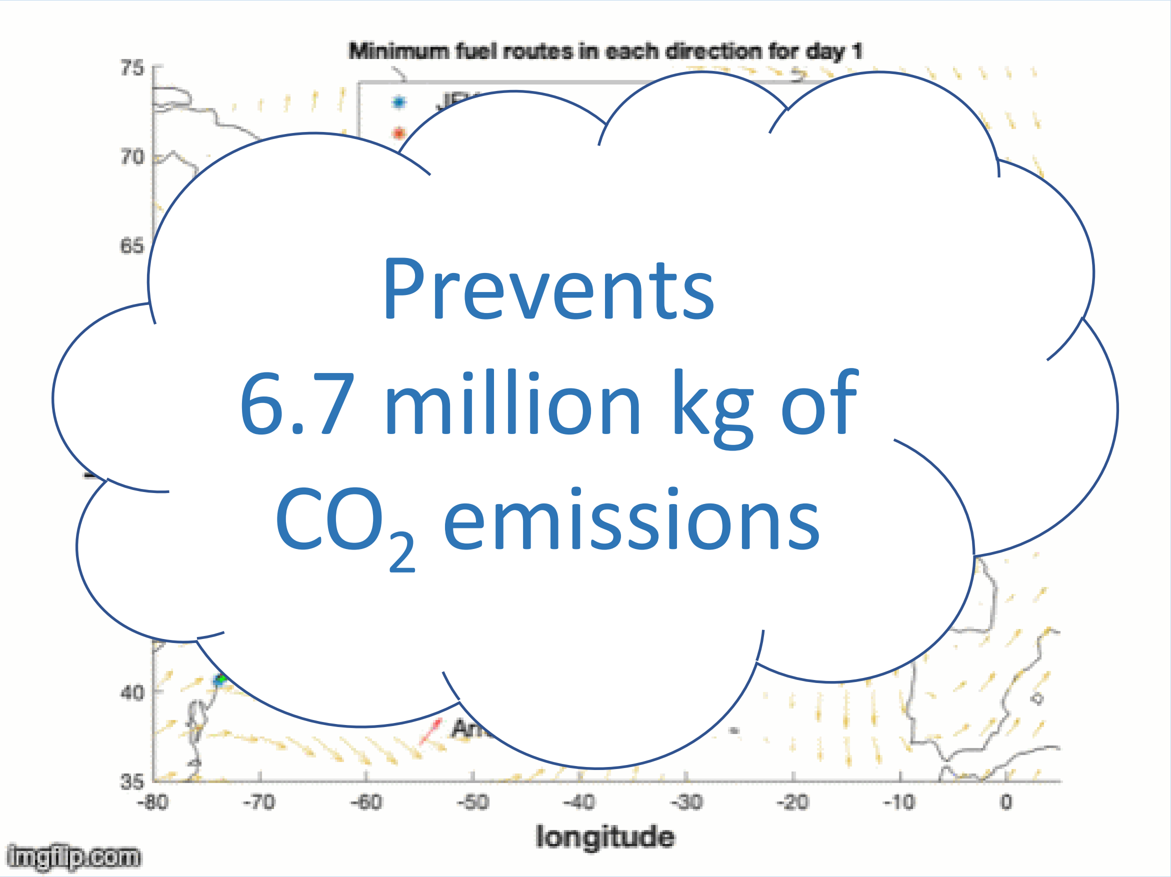


Prevents
6.7 million kg of
CO_2 emissions

cathie.wells@reading.ac.uk

Fixed time trajectories using a direct method:

Can fixed time flights be planned for a whole winter season to ensure that fuel is minimised?

$$g = \frac{W \times V_\infty}{\eta_0 \frac{L}{D} \times LCV}$$ (15,16)

In minimising fuel burn of transatlantic flights, can extra benefits result from controlling airspeed in addition to heading angle?

Involves:

- aircraft parameters
- ISA atmospheric parameters
- altitude
- airspeed
- temperature

cathie.wells@reading.ac.uk

Method: Assumptions

- Routes: JFK (40.6°N,73.8°W) and LHR (51.5°N,0.5°W).
- Winter period: 1st December, 2019 to 29th February, 2020.
- Winds and temperature from: National Center for Atmospheric Research
- 2.5° resolution.
- Constant pressure of 250 hPa equating approximately to FL340.
- Fixed time West 29 000 s, fixed time East 22 000 s.
- Mass of aircraft varies as fuel is burned.
- Fuel burn function based on recent papers by Poll and Schumann (15,16).

Optimal control problem 2

Parameters:

$$t_0 = 0$$
$$t_{final} = t_f$$

$$t \in [0, t_f] \subset \mathbb{R}$$

Boundary conditions:

$$\lambda(0) = \lambda_{\text{dept}}$$
$$\phi(0) = \phi_{\text{dept}}$$
$$M(0) = M_{\text{dept}}$$

State variables:

$$x_1 = \lambda(t)$$
$$x_2 = \phi(t)$$
$$x_3 = M(t)$$

$$\boldsymbol{x}(t) \colon [0, t_f] \longmapsto \mathbb{R}^3$$

Control variable:

$$\alpha(t) = \theta(t)$$

$$\alpha(t) \colon [0, t_f] \longmapsto \mathbb{R}$$

Optimal control problem 3

Parameters:

$$t_0 = 0$$
$$t_{final} = t_f$$

$$t \in [0, t_f] \subset \mathbb{R}$$

Boundary conditions:

$$\lambda(0) = \lambda_{\text{dept}}$$
$$\phi(0) = \phi_{\text{dept}}$$
$$M(0) = M_{\text{dept}}$$

State variables:

$$x_1 = \lambda(t)$$
$$x_2 = \phi(t)$$
$$x_3 = M(t)$$

$$\boldsymbol{x}(t): [0, t_f] \longmapsto \mathbb{R}^3$$

Control variables:

$$\alpha_1(t) = \theta$$
$$\alpha_2(t) = V$$

$$\boldsymbol{\alpha}(t): [0, t_f] \longmapsto \mathbb{R}^2$$

cathie.wells@reading.ac.uk

OCP 2 and 3:

Dynamical systems:

$$\dot{x}_1 = \frac{V\cos\alpha + \mathrm{u}(x_1, x_2)}{R\cos x_2}$$

$$\dot{x}_2 = \frac{V\sin\alpha + \mathrm{v}(x_1, x_2)}{R}$$

$$\dot{x}_3 = -g(x_1, x_2, x_3)$$

$$\dot{x}_1 = \frac{\alpha_2\cos\alpha_1 + \mathrm{u}(x_1, x_2)}{R\cos x_2}$$

$$\dot{x}_2 = \frac{\alpha_2\sin\alpha_1 + \mathrm{v}(x_1, x_2)}{R}$$

$$\dot{x}_3 = -g(x_1, x_2, x_3, \alpha_2)$$

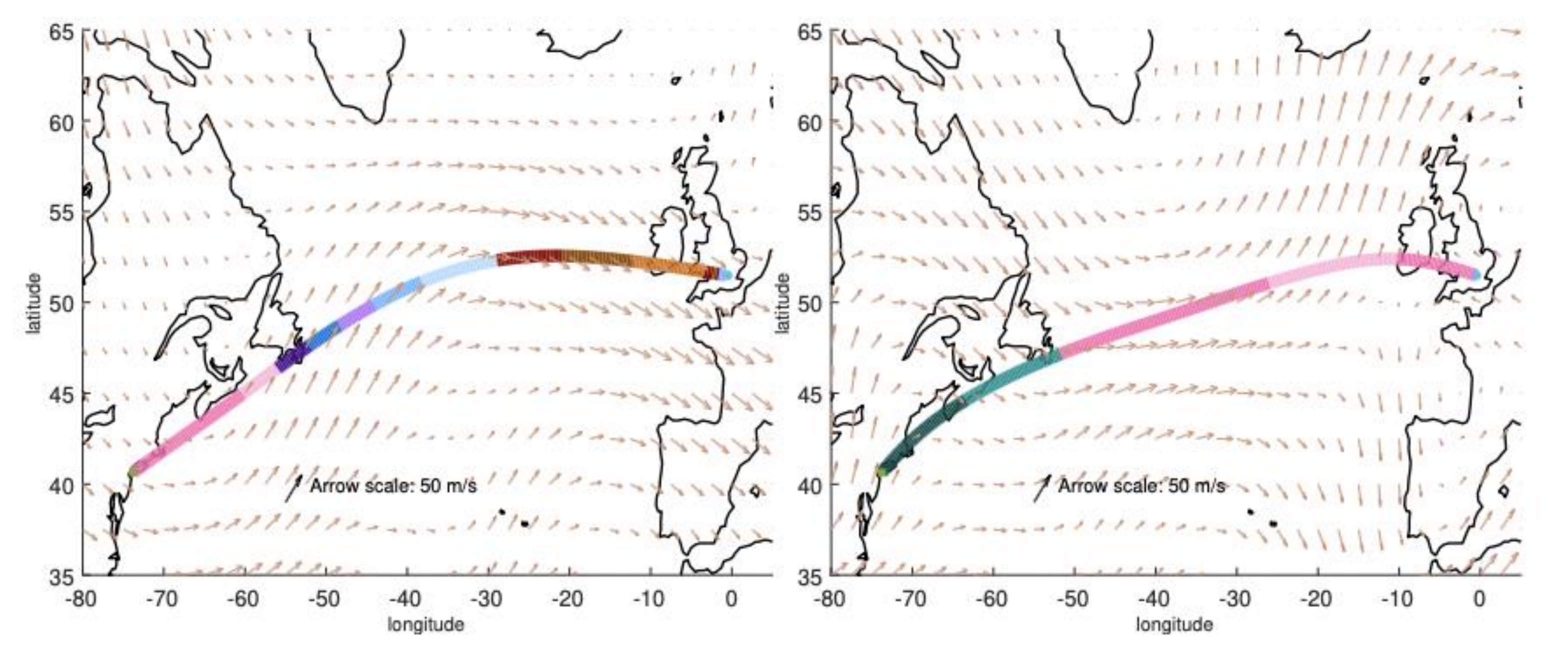
Cost functionals:

$$J(\boldsymbol{x}, \alpha) = \int_{t_0}^{t_{final}} g(\boldsymbol{x})\, dt$$

$$J(\boldsymbol{x}, \boldsymbol{\alpha}) = \int_{t_0}^{t_{final}} g(\boldsymbol{x}, \boldsymbol{\alpha_2})\, dt$$

Results: Air speed changes

Flying East:



(a) 28th January, 2020 Eastbound

(b) 25th January, 2020 Eastbound

Key to Airspeeds (V m/s) :

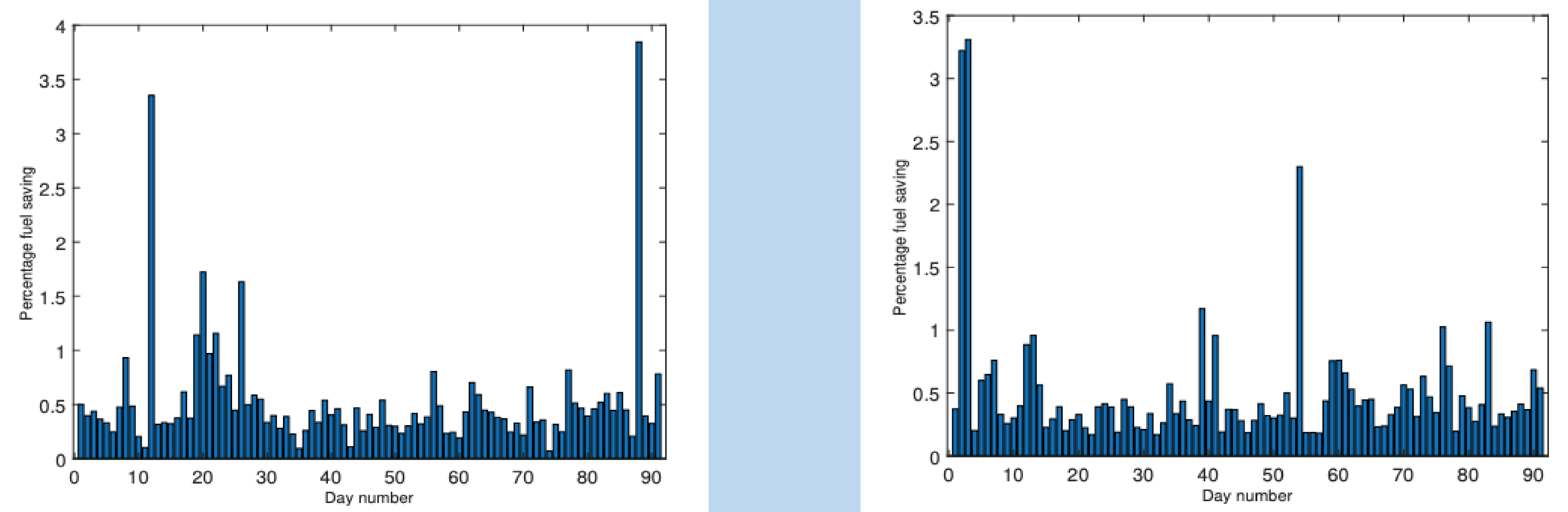
🟨 V≤ 201.5
🟩 201.5≤ V <204
🟧 204≤ V <206.5
🟫 206.5≤ V <209
🟥 209≤ V <211.5

🟦 211.5≤ V <214
🟦 214≤ V <216.5
🟪 216.5≤ V <219
🟦 219≤ V <221.5
🟪 221.5≤ V <224

■ 224≤ V <226.5
■ 226.5≤ V <229
■ 229≤ V <231.5
■ 231.5≤ V <234
■ 234≤ V <236.5

More tailwind along GCR, more variability in airspeed.

Overview of Findings:



(a) Savings made on each day West

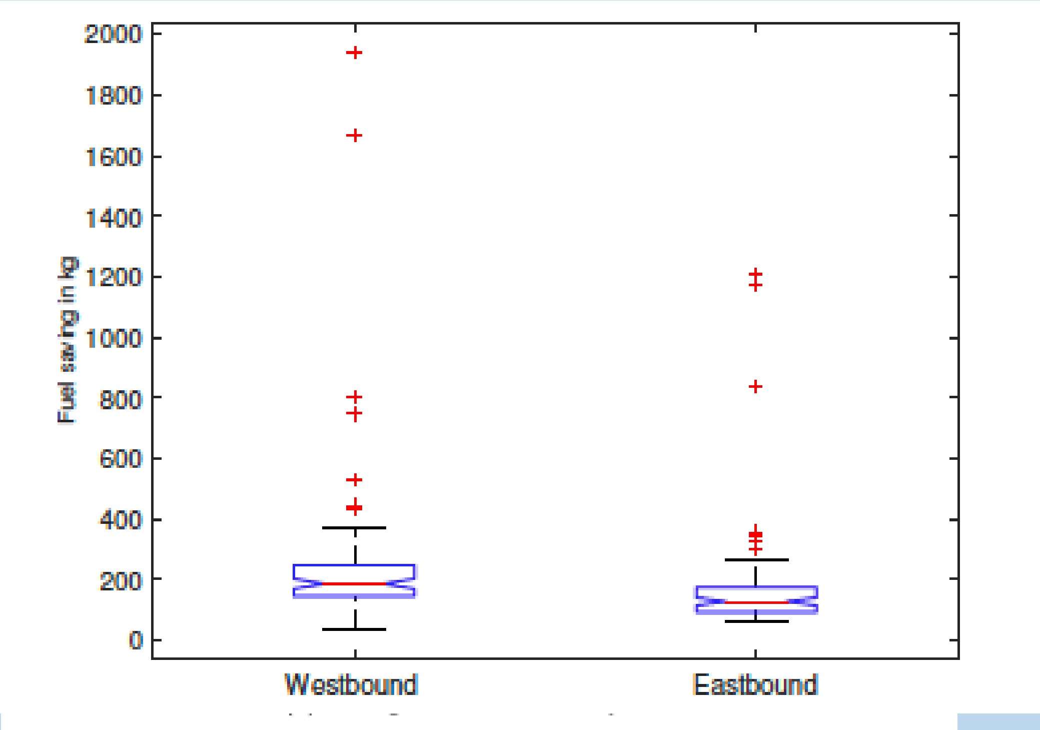
(b) Savings made on each day East

Daily variation in savings across winter season 2019-2020

cathie.wells@reading.ac.uk

Overview of Findings:

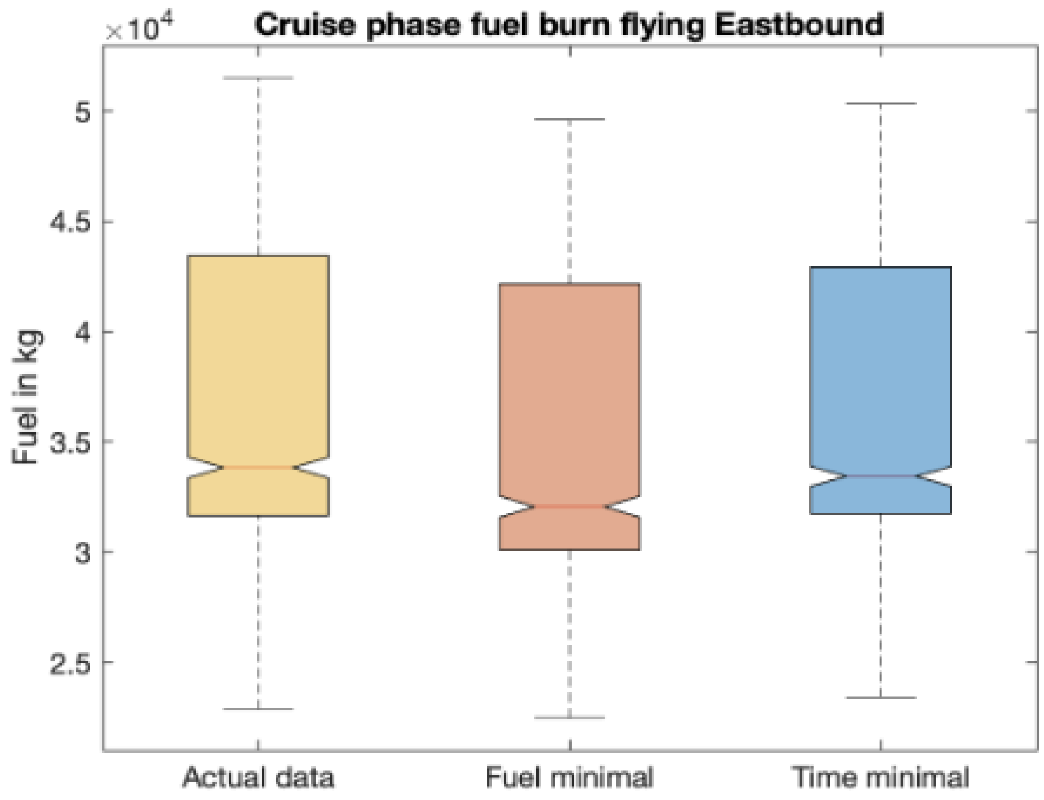
Prevents
an extra
723 000 kg of
CO_2 emissions



Fuel savings in kg made by controlling airspeed in addition to heading angle

Free time, fuel minimal trajectories using dynamic programming:

Would minimising fuel, but allowing time to be free, in a fixed altitude flight produce trajectories that are more fuel efficient than those currently flown and to what extent would the flight duration be changed?



cathie.wells@reading.ac.uk

Mathematics of Planet Earth

4th April, 2023

Optimal control problem 4 :

Parameters:

$$t_0 = 0$$
$$t_{final} = t_f \geq 0$$

$$t \in [0, t_f] \subset \mathbb{R}$$

Boundary conditions:

$$\lambda(0) = \lambda_{\text{dept}}$$
$$\phi(0) = \phi_{\text{dept}}$$
$$M(0) = M_{\text{dept}}$$

State variables:

$$x_1 = \lambda(t)$$
$$x_2 = \phi(t)$$
$$x_3 = M(t)$$

$$\boldsymbol{x}(t): [0, t_f] \longmapsto \mathbb{R}^2$$

Control variables:

$$\alpha_1(t) = \theta$$
$$\alpha_2(t) = V$$

$$\boldsymbol{\alpha}(t): [0, t_f] \longmapsto \mathbb{R}^2$$

Optimal control problem 5:

Parameters:

$$t_0 = 0$$
$$t_{final} = t_f \geq 0$$

$$t \in [0, t_f] \subset \mathbb{R}$$

Boundary conditions:

$$\lambda(0) = \lambda_{\text{dept}}$$
$$\phi(0) = \phi_{\text{dept}}$$

State variables:

$$x_1 = \lambda(t)$$
$$x_2 = \phi(t)$$

$$\boldsymbol{x}(t)\colon [0, t_f] \longmapsto \mathbb{R}^2$$

Control variables:

$$\alpha_1(t) = \theta$$
$$\alpha_2(t) = V$$

$$\boldsymbol{\alpha}(t)\colon [0, t_f] \longmapsto \mathbb{R}^2$$

Optimal control problem 4:

Dynamical system:

$$\dot{x}_1 = \frac{\alpha_2 \cos \alpha_1 + \mathrm{u}(x_1, x_2)}{R \cos x_2}$$

$$\dot{x}_2 = \frac{\alpha_2 \sin \alpha_1 + \mathrm{v}(x_1, x_2)}{R}$$

$$\dot{x}_3 = -g$$

Cost functional:

$$J(\boldsymbol{x}, \boldsymbol{\alpha}) = \int_{t_0}^{t_{final}} L(\boldsymbol{x}, \boldsymbol{\alpha}) dt = \int_{t_0}^{t_{final}} g(\boldsymbol{x}, \boldsymbol{\alpha_2})\, dt$$

$$J(t): [0, t_f] \times \mathbb{R}^3 \times \mathbb{R} \longmapsto \mathbb{R}$$

Optimal control problem 5:

Dynamical system:

$$\dot{x}_1 = \frac{\alpha_2 \cos \alpha_1 + \mathrm{u}(x_1, x_2)}{R \cos x_2}$$
$$\dot{x}_2 = \frac{\alpha_2 \sin \alpha_1 + \mathrm{v}(x_1, x_2)}{R}$$

Cost functional:

$$J(\boldsymbol{x}, \boldsymbol{\alpha}) = \int_{t_0}^{t_{final}} L(\boldsymbol{x}, \boldsymbol{\alpha}) dt = \int_{t_0}^{t_{final}} 1 \, dt$$

$$J(t) \colon [0, t_f] \times \mathbb{R}^2 \times \mathbb{R} \longmapsto \mathbb{R}$$

cathie.wells@reading.ac.uk

4th April, 2023

Dynamic programming: [17]

- ➢ Solve the Hamilton Jacobi Bellman Equation:

Time minimal:

$$\begin{cases} v(x) + \sup\limits_{\alpha \in \mathcal{A}} \{-Dv \cdot f(x,\alpha) - 1\} = 0 & x \in \mathbb{R}^N \setminus C \\ v(x) = 0 & \forall x \in C \end{cases}$$

$$v(x) := \begin{cases} 1 & T(x) = +\infty \\ 1 - e^{-T(x)} & T(x) < +\infty \end{cases}$$

Fuel minimal:

$$\begin{cases} V(x) - \inf\limits_{\alpha \in \mathcal{A}} \{g(x,\alpha) + f(x,\alpha) \cdot \Delta V(x) - (g(x,\alpha) - 1)V(x)\} = 0 & x \in \mathbb{R}^n \setminus C \\ V(x) = 0 & x \in C \end{cases}$$

- ➢ Use value function map to find optimal feedback control

cathie.wells@reading.ac.uk

Semi-Lagrangian Scheme: (time) [18]

Updated value for space node i and iteration m.

I represents linear interpolation of value function outside of grid points.

$$[V]_i^{m+1} = \min_{a \in \mathcal{A}} \{ e^{-\Delta t} I[V]_i^m (y_i + \Delta t f(y_i, a)) \} + 1 - e^{-\Delta t}$$

Value is calculated for all combinations of control variable and minimum accepted as the node value.

Here the dynamical system is approximated by an Euler scheme with time step Δt .

These terms are derived from the Kružkow transform which can be expressed as:

$$V^{\Delta t}(x) := \begin{cases} 1 & T(x) = +\infty \\ 1 - e^{-\Delta t N(x)} & \text{else} \end{cases}$$

c.a.wells@pgr.reading.ac.uk

Semi-Lagrangian Scheme: (fuel)[17]

Updated value for space node i and iteration m.

I represents linear interpolation of value function outside of grid points.

The extra term is included to account for the fact that the running cost is no longer 1 as in the time minimal version.

$$[V]_i^{m+1} = \min_{a\in\mathcal{A}}\{I[V]_i^m(y_i + \Delta t f(y_i, a)) + \Delta t g(y_i, \alpha)(1 - V_i^m(y_i))\}$$

Value is calculated for all combinations of control variable and minimum accepted as the node value.

The fuel burn function g is now included in the formulation.

Algorithm to find value map:

Estimate starting value at grid point:
0 for target
1 elsewhere

Construct the interpolant: $I[V^m]$

Find V^{m+1} for each combination of controls. If $y_i + \Delta t f(y_i, a)$ lies outside of the specified grid, set the value for this point to 1

Find minimum result and store

Measure distance $\|V^{m+1} - V^m\|$

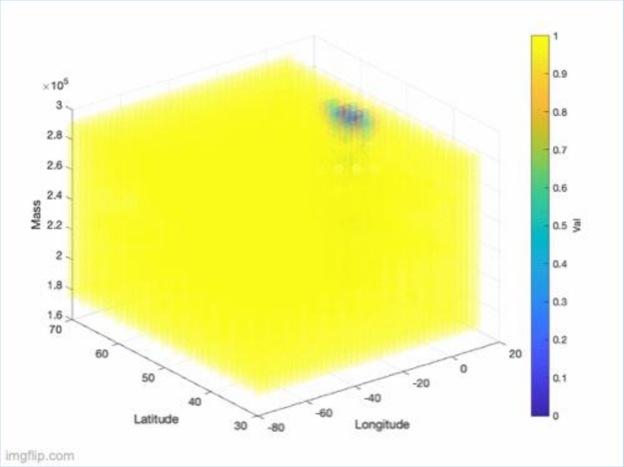
≥ tolerance

Update $V^m = V^{m+1}$

Reset target values to 0

< tolerance

Stop.
Move to next grid point

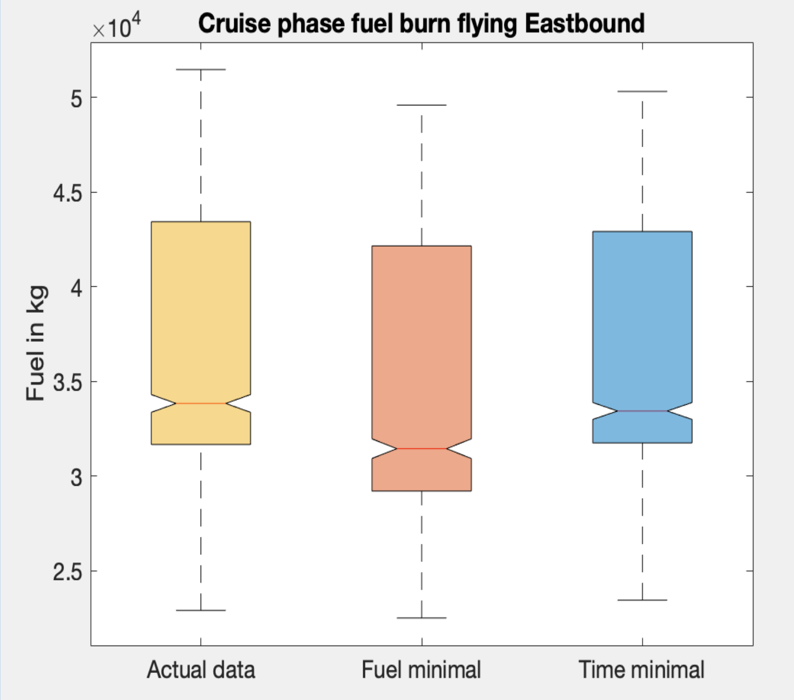


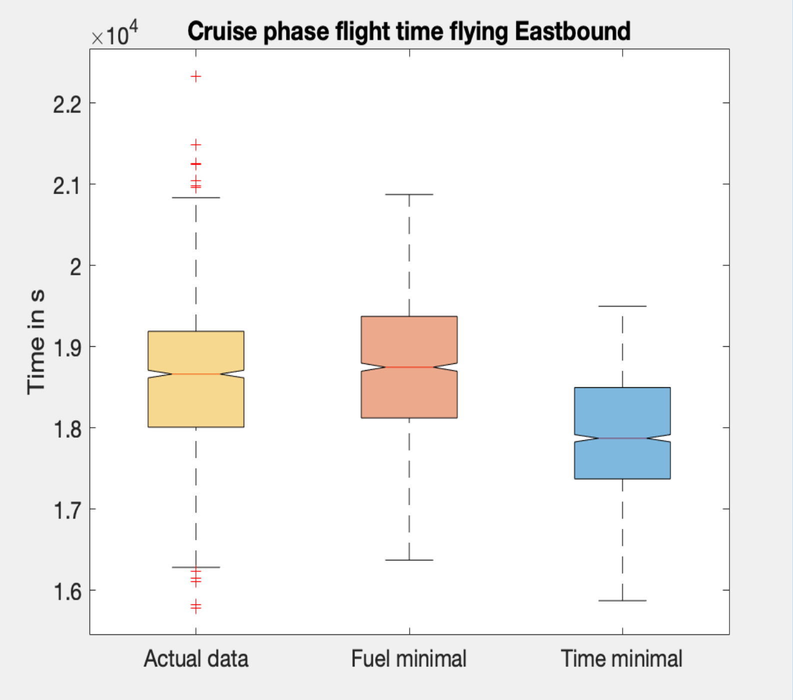
cathie.wells@reading.ac.uk

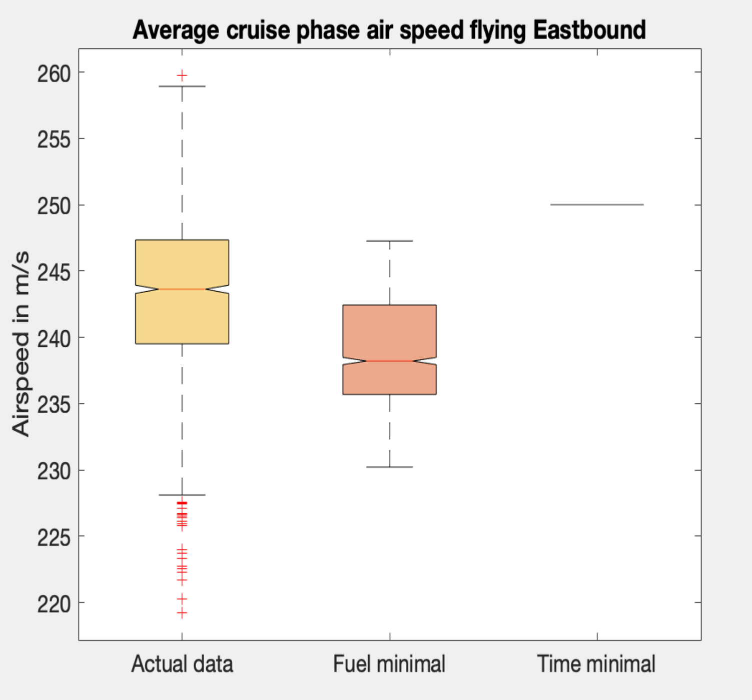
Actual flight data

- Positions and times recorded by flightradar24.com.
- Using wind field and ground speed, airspeed can be recovered.
- If outside practical bounds, adjusted and time altered accordingly.
- Altitude kept to approximately FL 340 (barometric pressure of 250 hPa).
- Fuel burn calculated as for simulated flights.
- Flights from : American Airlines
  British Airways
  Delta Air Lines
  Virgin Atlantic
- 1547 eastbound and 1567 westbound flights considered.

Results: comparison across all data



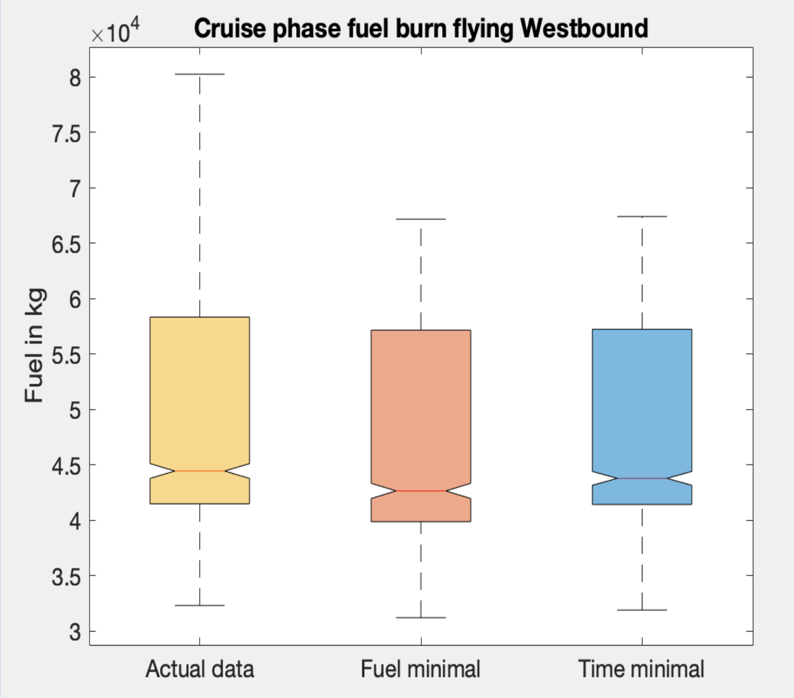


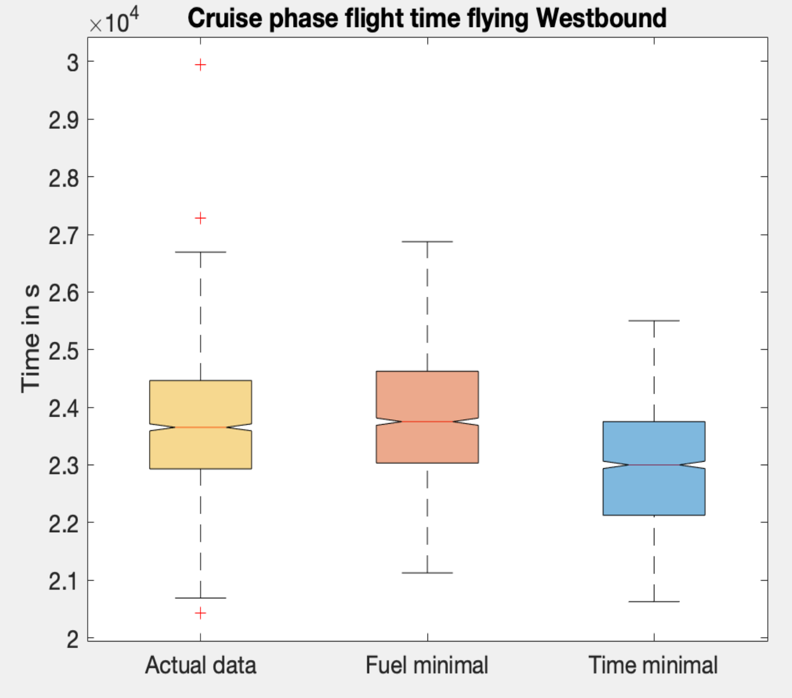


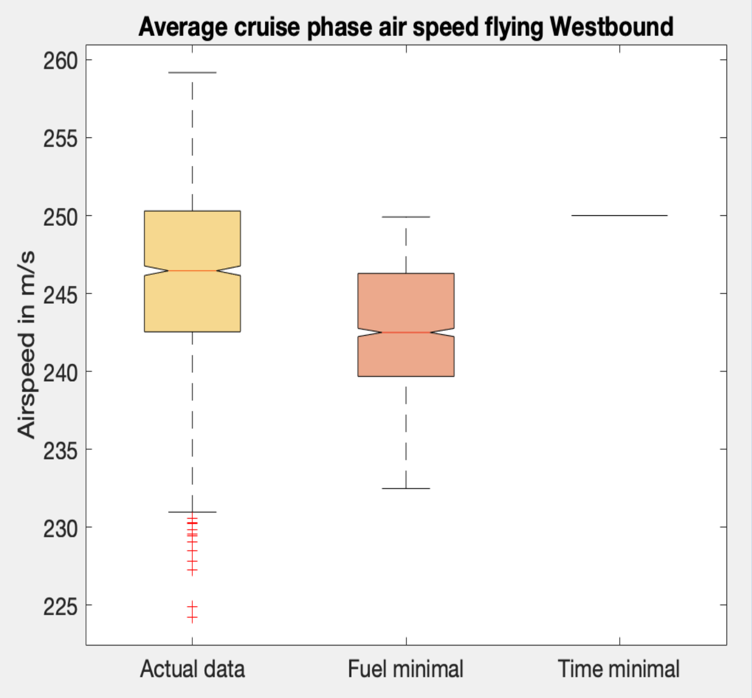
At 5% level: significantly less fuel used, but time of flight not significantly longer. Significantly lower average airspeeds for flights.

cathie.wells@reading.ac.uk

Results: comparison across all data


Cruise phase fuel burn flying Westbound

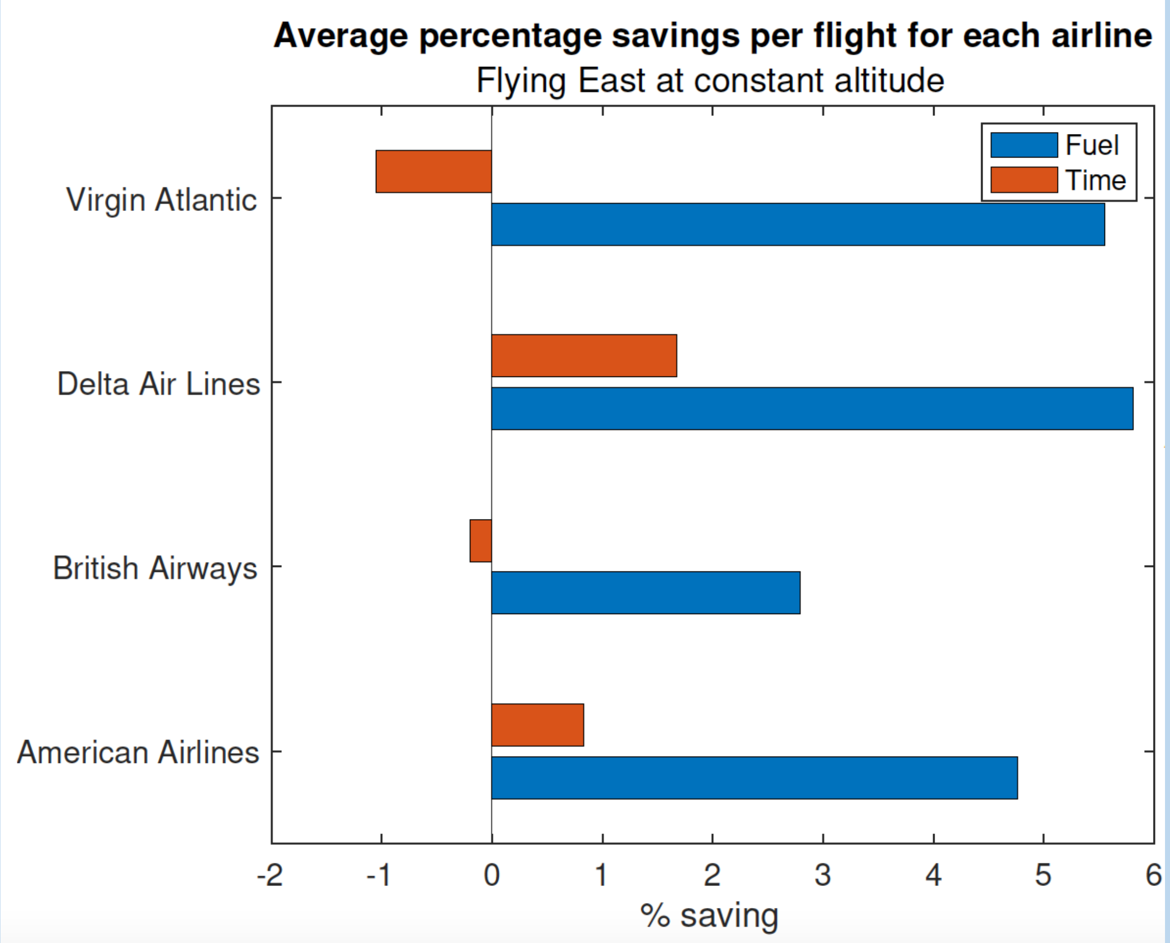

Cruise phase flight time flying Westbound

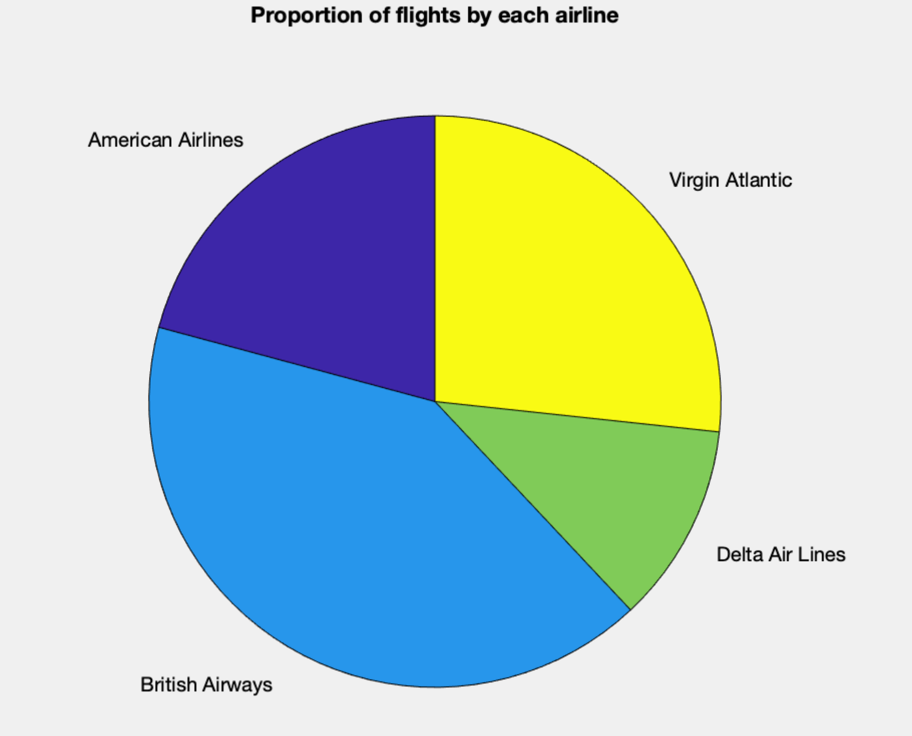

Average cruise phase air speed flying Westbound

Similar to Eastbound results.

Slightly reduced fuel savings and not such a disparity in airspeed.

Results: differences between airlines





Overview of free-time, fuel minimal results:

- Total fuel reduction Eastbound: 2.5 million kg
- Total fuel reduction Westbound: 2.8 million kg
- Each 1kg of aviation fuel burned produces 3.16 kg of CO_2
- Across this one winter period :

16.6 million kg reduction in CO_2 emissions

1 307 average UK residents

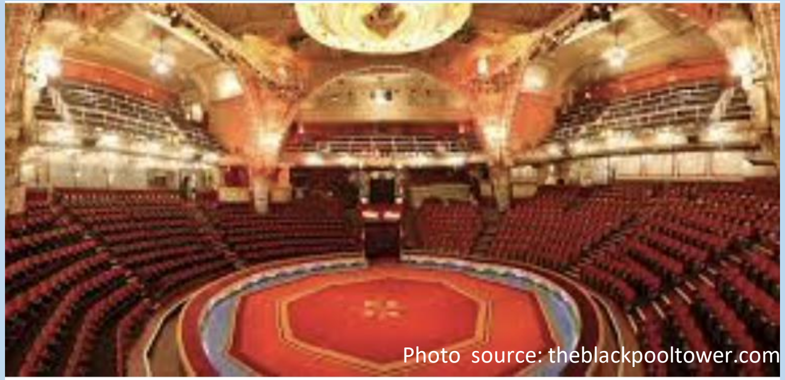


Photo source: theblackpooltower.com

15 962 average Pakistan residents

Photo by Anjum Naveed

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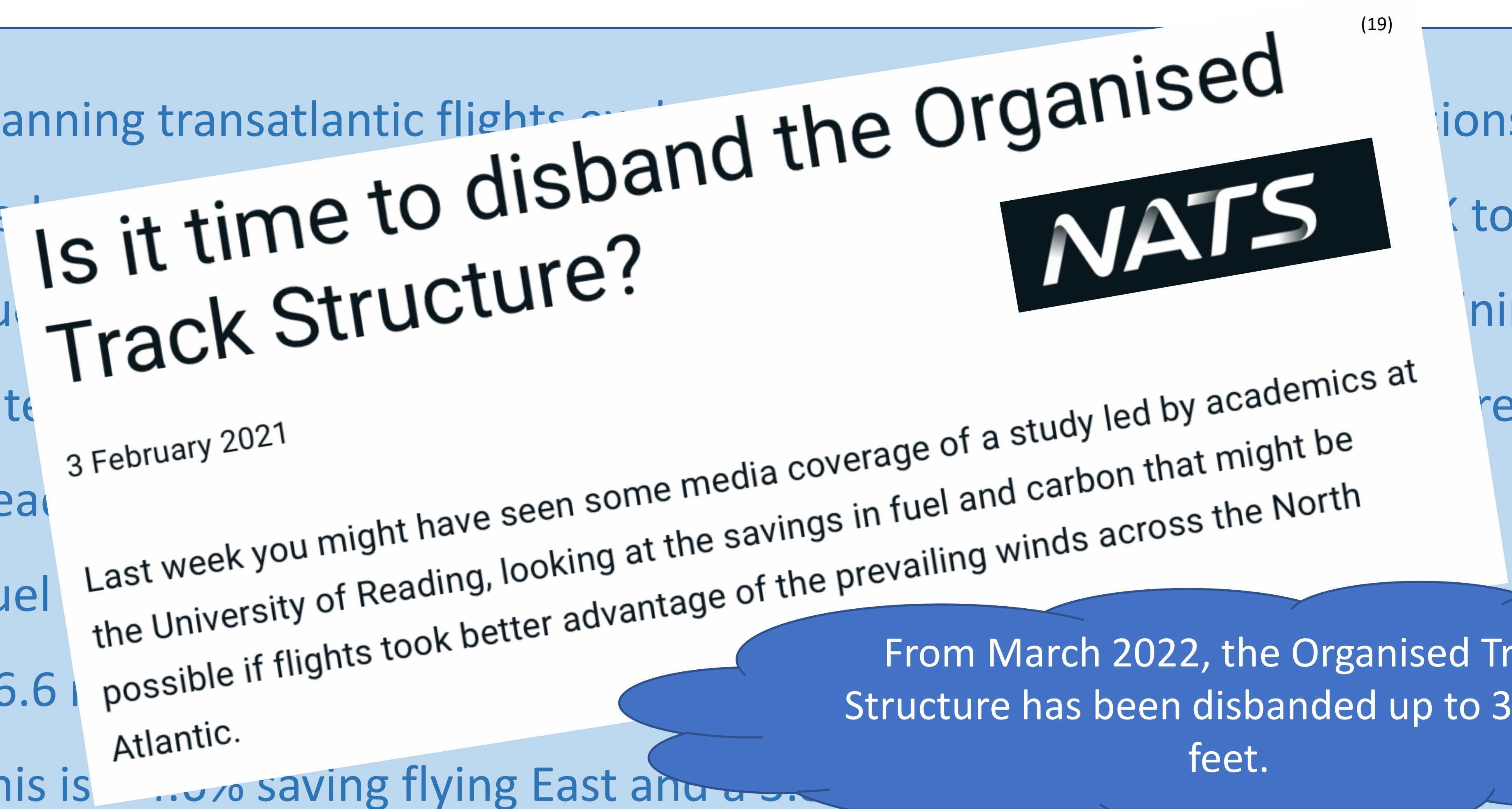
Conclusions:

- Planning transatlantic flights exploiting available winds reduces emissions.
- Reduction of 6.7 million kg of CO_2 emissions on the route between JFK to LHR one winter [20].
- Fuel minimisation for a fixed time route may be more practical than minimising time.
- Altering airspeed can save an extra 723 000 kg of CO_2 emissions compared with just altering heading for fixed-time, fuel minimal routes [21].
- Fuel minimal free-time flights compared with actual flight data:

  16.6 million kg CO_2 emissions saved.
- This is a 4.6% saving flying East and a 3.8% saving flying West.

Conclusions:

- Planning transatlantic flights ov ions.
- Re to LHR one winter [20].
- Fu nimising time.
- Alte red with just altering hea
- Fuel 16.6
- This is .0% saving flying East and a 3.



(19)

Is it time to disband the Organised Track Structure?

NATS

3 February 2021

Last week you might have seen some media coverage of a study led by academics at the University of Reading, looking at the savings in fuel and carbon that might be possible if flights took better advantage of the prevailing winds across the North Atlantic.

From March 2022, the Organised Track Structure has been disbanded up to 33 000 feet.

New Horizons:

- Controlling cruise altitude.
- Including climb and descent phases.
- Avoiding areas of convection and/or super saturated icy regions.
- Training neural networks for faster rerouting as new information becomes available.

Results so far published in Environmental Research Letters (20), Optimization and Engineering (21) and Transportation Research Part D (Transport and Environment) (22).

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